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The Projection Operator Approach to the Fokker–Planck Equation. II. Dichotomic and Nonlinear Gaussian Noise

Sandro Faetti,[1,2] Leone Fronzoni,[1,2] Paolo Grigolini,[1,2,3] Vincenzo Palleschi,[3,4] and Girolamo Tropiano[1,4]

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Two models for the Freedericksz transition in a fluctuating magnetic field are considered: one is based on a dichotomic and the other on a nonlinear Gaussian noise. Both noises are characterized by a finite correlation time τ. It is shown that the linear response assumption leading to the "best Fokker–Planck approximation" in the dichotomic and nonlinear Gaussian cases can be trusted only up to the order τ^1 and τ^0, respectively. The role of the corrections to the linear response approximation is discussed and it is shown how to replace the non-Fokker–Planck terms stemming from these corrections with equivalent terms of standard type. This technique is shown to produce perfect agreement with the exact analytical results (dichotomic noise) and to satisfactorily fit the results of analog simulation (nonlinear Gaussian noise).

KEY WORDS: Breakdown of the Fokker–Planck structure; dichotomic noise; nonlinear Gaussian noise; Freedericksz transition.

1. INTRODUCTION

It has been shown[1] that the so-called best Fokker–Planck approximation (BFPA) theory[2] relies on the linear response approximation.[3,4] When applied to study the energy absorption from a radiation field this approximation is shown[1] to express the energy absorbed per unit of time

[1] Dipartimento di Fisica dell' Universita' di Pisa, Pisa, Italy.
[2] Gruppo Nazionale di Struttura della Materia del CNR, Pisa, Italy.
[3] Institute for Nonlinear Science, University of California-San Diego, La Jolla, California 92093.
[4] Istituto di Fisica Atomica e Molecolare del CNR, Pisa, Italy.

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in terms of the equilibrium velocity autocorrelation function precisely in the same way as the Kubo theory.[5] On the other hand, the Kubo theory,[5] albeit severely criticized by van Kampen,[6] has a wide dominion of application and is currently applied to interpreting experimental results in many research areas. This means that the BFPA should be regarded as being a well-founded theory: in the special case where the excitation source is provided by a noise with a finite correlation time τ, the range of validity of the BFPA should be from $\tau = 0$ to $\tau = \infty$, provided that the intensity of noise is kept weak enough.[7,8] However, in ref. 2 a system described by the equation

$$\dot{x} = \varphi(x) + \psi(x)\,\xi(t) \tag{1.1}$$

where $\xi(t)$ is a Gaussian noise with vanishing mean value and correlation time τ, the BFPA was shown to break down at the second order in τ. This is a consequence of the nonlinear nature of the systems studied in ref. 2. From the projection method of ref. 2 it is also evident that the second order in the interaction between x and ξ is totally unaffected by the statistics of ξ. To realize that its breakdown depends on the nonlinear nature of the system (otherwise the linear nature of the system and the Gaussian statistics of the noise will lead to an exact result), it is necessary to determine the corrections to the BFPA at the fourth order in the interaction between x and ξ.

In this paper we study the corrections tot he BFPA generated by noise with non-Gaussian statistics. To this purpose we study two models for the Freedericksz transition in a fluctuating magnetic field,[9,10] which are described by equations with the same structure as Eq. (1.1) but with non-Gaussian statistics. The breakdown of the BFPA is shown to be still more severe than in the Gaussian case.

We stress the decisive role played in this paper by the study of the dichotomic case. This is an especially illuminating case since it is accompanied by an exact analytical solution, to which the predictions of our approximate theory can be compared without the usual concern on the precision of the comparison data when these are provided via analog or digital simulation. Nevertheless, the analog simulation will be used for the auxiliary purpose of illustrating the breakdown of the BFPA equation in the appealing case of nonlinear Gaussian noise (where no exact solution is available).

The outline of the paper is as follows. Section 2 is devolted to illustrating two cases of breakdown of the BFPA equation. Section 3 shows how to derive the first nonvanishing corrections to the BFPA equation for the two cases of noise statistics here under investigation. In Section 4 we replace nonstandard diffusion terms stemming from the corrections to the

BFPA with equivalent standard diffusion operators (using the same procedure as in ref. 2). Concluding remarks are found in Section 4. The Appendix is devoted to illustrating the analog apparatus used to study the nonlinear Gaussian case.

2. THE BREAKDOWN OF THE FOKKER–PLANCK EQUATION

The major aim of the present section is to illustrate the breakdown of the BFPA in two different cases. In the first case (dichotomic noise) the exact solution of the BFPA equation is compared to the exact equilibrium distribution of the variable x which accompanies the dichotomic statistics. In the latter case (nonlinear Gaussian noise) this breakdown is illustrated by comparing the exact solution of the BFPA equation to the results of analog simulation (see the Appendix for a description of the corresponding experimental setup). Note that analog simulation is of vital importance to study the latter case, where no exact analytical solution is available. The accuracy of the analog simulation apparatus is checked by applying it to the case of dichotomic noice and comparing the corresponding result with the exact analytical solution provided in this case by Horsthemke *et al.*[9] (see Fig. 1). Note that no fitting parameters but the normalization constant is used to get this excellent agreement.

The theoretical reasons for the breakdown of the BFPA equation under these two different circumstances will be analyzed in later sections.

2.1. A Model for the Freedericksz Transition in a Fluctuating Magnetic Field: Dichotomic Case[9]

The interesting phenomenon of the Freedericksz transition in a fluctuating magnetic field has been recently modeled by two groups of investigators[9,10] by using virtually the same stochastic differential equation,[5] which reads

$$\dot{x} = -\Gamma x + [h - \eta(t)]^2 (\alpha x - \beta x^3) \tag{2.1}$$

where $\eta(t)$ is a noice characterized by

$$\begin{aligned} \langle \eta(t) \rangle &= 0 \\ \langle \eta(t_1)\, \eta(t_2) \rangle &= \delta^2 \exp(-\gamma\, |t_2 - t_1|) \end{aligned} \tag{2.2}$$

[5] Note that Sagues and San Miguel[10] studied the case where the variable x is also driven by an additive stochastic force. Within the context of the present paper we limit ourselves to the case where this additive stochastic force vanishes and the analytical results of Sagues and San Miguel will be referred to as adapted to this special case.

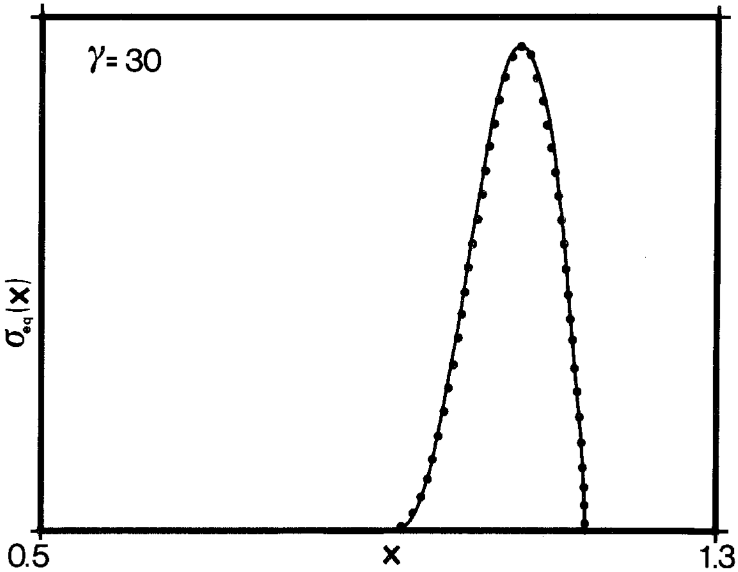


Fig. 1. Equilibrium distribution of the variable x in the case of the dichotomic noise (see Section 2.1). (—) The exact equilibrium distribution [Eq. (2.4)] of the system described by Eq. (2.1). (●) The results of the analog simulation. The normalization constant was obtained by matching theory to the measurements at the maximum. The parameters of Eq. (2.1) are $\Gamma = 1$, $\gamma = 30$, $h = 1.5$, $\langle \eta^2 \rangle = 0.04$, $\alpha = 1$, and $\beta = 1/2$.

Note that in the original models[9,10] α and β derive from the expansion of $\sin x$ (x denoting an angle variable), leading to $\alpha = 1$ and $\beta = 1/2$.

Our interest in Eq. (2.1) is motivated by the fact that this equation can be given the same form as the stochastic differential equation (1.1) (which, in turn, is precisely the same as that studied in the companion paper[2]). Indeed, Eq. (2.1) is equivalent to

$$\dot{x} = \varphi(x) + \psi(x)\,\xi(t) \tag{2.1'}$$

where

$$\xi(t) \equiv \eta^2(t) - \langle \eta^2(t) \rangle - 2h\eta(t) \tag{2.3a}$$

$$\varphi(x) \equiv Ax - Bx^3 \tag{2.3b}$$

$$A \equiv [h^2 + \langle \eta^2(t) \rangle]\alpha - \Gamma \tag{2.3c}$$

$$B \equiv [h^2 + \langle \eta^2(t)]\beta \tag{2.3d}$$

$$\psi(x) \equiv \alpha x - \beta x^3 \tag{2.3e}$$

In the dichotomic case

$$\eta^2(t) \equiv \langle \eta^2(t) \rangle \tag{2.3f}$$

and the Fokker–Planck equation associated with Eq. (2.1) can be given the exact equilibrium distribution[9]

$$\sigma_{\text{eq}}^{(x)} = N \frac{\psi(x)}{4h^2 \delta^2 \psi^2(x) - \varphi^2(x)} \otimes \left[\frac{x^2}{x_+^2 - x^2}\right]^{-\gamma/4(A+2h\alpha\delta)} \otimes \left[\frac{x^2}{x^2 - x_-^2}\right]^{-\gamma/4(A-2h\alpha\delta)} \tag{2.4}$$

where

$$x_+^2 \equiv \frac{A + 2h\alpha\,\delta}{B + 2h\beta\,\delta}, \qquad x_-^2 \equiv \frac{A - 2h\alpha\,\delta}{B - 2h\beta\,\delta}, \qquad \delta \equiv \langle \eta^2(t) \rangle^{1/2} \tag{2.5}$$

The support[9] of the equilibrium distribution $\sigma_{\text{eq}}(x)$ is

$$\mathfrak{U} = \begin{array}{ll} \{0\} & \text{if } x_-^2 < 0, \quad x_+^2 < 0 \\ \{0, x_+\} & \text{if } x_-^2 < 0, \quad x_+^2 > 0 \\ \{x_-, x_+\} & \text{if } x_-^2 > 0, \quad x_+^2 > 0 \end{array} \tag{2.5$'$}$$

Note that in Section 4 this analytical equilibrium distribution is rewritten in a form more convenient for the theoretical purposes of that section [see Eq. (4.1)].

The projection approach to the Fokker–Planck equation at the second order in $\mathscr{L}_1$ does not rely on the Gaussian assumption.[2] This means that even in the present case at the second order in $\mathscr{L}_1$ we obtain the BFPA

$$\frac{\partial}{\partial t}\sigma(x; t) = \left[-\frac{\partial}{\partial x}\varphi(x) + \frac{4h^2\langle \eta^2 \rangle}{\gamma} \times \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\Phi(x)\right]\sigma(x; t) \tag{2.6}$$

where $\Phi(x)$ is shown[1,6,7] to satisfy the differential equation

$$\psi(x) = \Phi(x) + \tau[\varphi(x)\,\Phi'(x) - \varphi'(x)\,\Phi(x)] \tag{2.7}$$

$\tau \equiv 1/\gamma$ denotes the noise correlation time. It is easy to show that Eq. (2.7) is satisfied by a solution with the form $\Phi(x) = Rx - R|x^3$. By replacing this expression Eq. (2.7) with $\varphi(x)$ and $\psi(x)$ provided by Eqs. (2.3b) and (2.3e), we can determine the unknown coefficients R and R'. The resulting expression for $\Phi(x)$ is

$$\Phi(x) = \psi(x) + kx^3/(2A + \gamma) \tag{2.8}$$

where

$$k \equiv 2(A\beta - B\alpha) \tag{2.8'}$$

The white noise approximation leads to

$$\Phi(x) = \psi(x) \tag{2.9}$$

In the limit $\tau \to 0$, the BFPA, the white noise approximation, and the exact result virtually coincide. On the other hand, Fig. 2 shows that, for τ finite, the white noise approximation provides a result closer to the exact equilibrium distribution than the predictions of the BFPA equation. This is a severe violation of the BFPA (a still more severe one will be illustrated with theoretical arguments and analog simulation in the case of nonlinear Gaussian noise). We plan to show that the breakdown of the BFPA in the cases of dichotomic and nonlinear Gaussian noise is more severe than in the Gaussian case. Whereas in the Gaussian case[2] it takes at the second order in τ, in the case of the dichotomic and nonlinear Gaussian noise it takes place at the first and zeroth orders in τ, respectively.

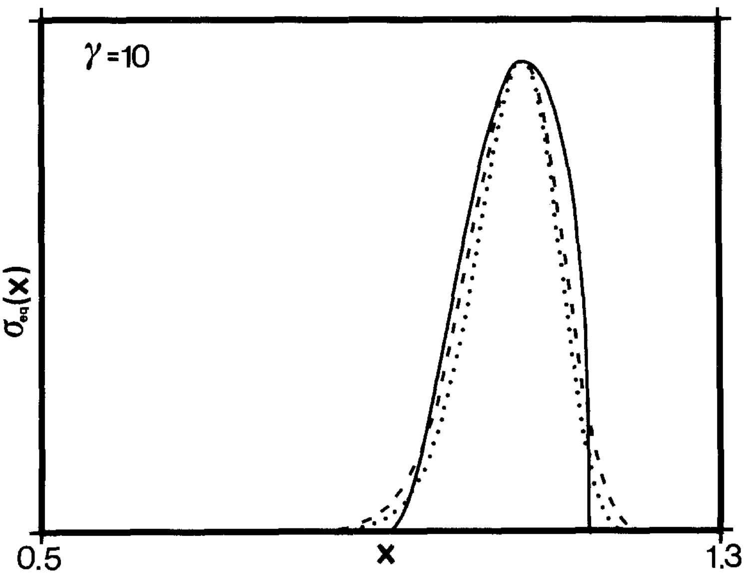


Fig. 2. Theoretical equilibrium distributions of the variable x in the dichotomic case of Section 2.1. (—) The exact theoretical result of Eq. (2.4). ($\cdots$) The approximate equilibrium distribution resulting from the BFPA equation (2.6), i.e., the equilibrium distribution of Eq. (4.3) with Φ_{eff} replaced by Φ of Eq. (4.5). (- -) The result corresponding to the white noise approximation [Eq. (2.9)]. Note that the approximate procedure of Section 4.1 provides an equilibrium distribution which coincides with the exact result (solid line). The parameters of Eq. (2.1) are $\Gamma = 1$, $\gamma = 10$, $h = 1.5$, $\langle \eta^2 \rangle = 0.04$, $\alpha = 1$, and $\beta = 1/2$.

2.2. A Model for the Freedericksz Transition in a Fluctuating Magnetic Field: Nonlinear Gaussian Noise[11]

When $\eta(t)$ of Eq. (2.2) is a Gaussian noise the simplifying condition of Eq. (2.3f) can no longer be used. The resulting nonlinear Gaussian noise $\xi(t)$ is characterized by the biexponential correlation function:

$$\langle \xi(0)\, \xi(t) \rangle = 2\langle \eta^2 \rangle^2 e^{-2\gamma t} + 4h^2 \langle \eta^2 \rangle e^{-\gamma t} \tag{2.10}$$

This does not result in any technical difficulty, since it is a straightforward matter to apply the theory of the companion paper[2] to the general case of the multiexponential correlation function:

$$\langle \xi(0)\, \xi(t) \rangle = \sum_{i=1}^{N} \langle \xi_i^2 \rangle e^{-\gamma_i t} \tag{2.11}$$

It is immediately seen that the BFPA in this case reads

$$\frac{\partial}{\partial t} \sigma(x; t) = \left[-\frac{\partial}{\partial x} \varphi(x) + \sum_{i=1}^{N} D_i \times \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \Phi_i(x) \right] \sigma(x; t) \tag{2.12}$$

where

$$D_i \equiv \langle \xi_i^2 \rangle / \gamma_i, \qquad i = 1, N \tag{2.13}$$

and the functions $\Phi_i(x)$ satisfy the differential equations

$$\psi(x) = \Phi_i(x) + \tau_i [\varphi(x)\, \Phi_i'(x) - \varphi'(x)\, \Phi_i(x)], \quad i = 1, N \tag{2.14}$$

where

$$\tau_i \equiv 1/\gamma_i, \qquad i = 1, N \tag{2.15}$$

In the nonlinear Gaussian case associated with Eq. (2.10) we have to deal with two differential equations [of the type of Eq. (2.14)], i.e., $N = 2$ and

$$\gamma_1 = 2\gamma; \quad \gamma_2 = \gamma; \quad \langle \xi_1^2 \rangle = 2\langle \eta^2 \rangle^2; \quad \langle \xi_2^2 \rangle = 4h^2 \langle \eta^2 \rangle \tag{2.15'}$$

Each equation of this bidimensional set can be solved with the same approach as that used in the preceding subsection. We thus obtain

$$\Phi_1(x) = \psi(x) + kx^3/2(A + \gamma) \tag{2.16}$$

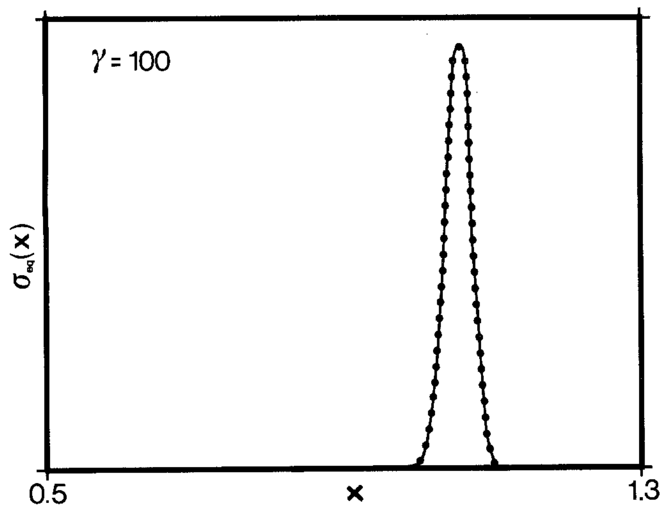
$$\Phi_2(x) = \psi(x) + kx^3/(2A + \gamma) \tag{2.16'}$$

In the white noise limit ($\gamma \to \infty$) we are immediately led to

$$\Phi_1(x) = \Phi_2(x) = \psi(x) \tag{2.17}$$

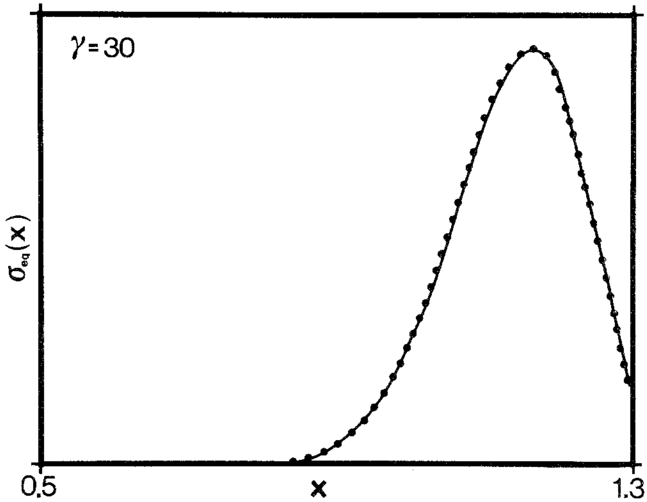
which, in turn, when replaced into Eq. (2.12) with $N = 2$ produces the same Fokker-Planck equation as that recently used by Sagues and San Miguel[9] to study the Freedericksz transition in a fluctuating magnetic field. From Eq. (2.16) it appears that in this case the BFPA equation only affects the cubic terms of $\Phi_1(x)$ and $\Phi_2(x)$, thereby preventing the color of noise from affecting the threshold value for the Freedericksz transition (for the corresponding definition of the threshold see, for instance, ref. 9).

Figure 3a shows that at small values of both τ and noise intensity the BFPA equation, the white noise approximation,[10] and analog simulation provide virtually coincident results. Note that the solid curve denoting the

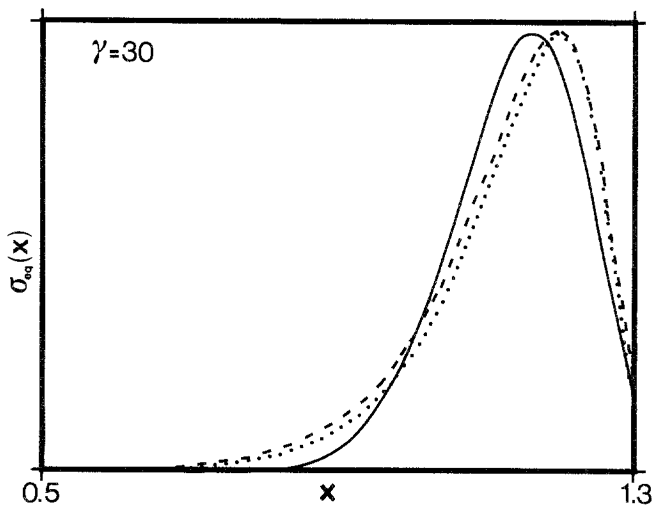


(a)

Fig. 3. Comparison of (●) the results of analog simulation with the equilibrium distribution predicted by (- -) the white noise approximation of ref. 10, (···) the BFPA equation, and (—) our theoretical approach (see Sections 3 and 4) in the case of nonlinear Gaussian noise. The normalization constants of all these distributions were obtained by matching the value of $\sigma_{eq}(x)$ at the maxima. All cases are characterized by the parameters $\Gamma = 1$, $h = 1.5$, $\alpha = 1$, and $\beta = 1/2$. (a) $\gamma = 100$, and $\langle \eta^2 \rangle = 0.04$. Note that in this case the three curves are indistinguishable form one another. (b) Comparison between our theoretical result and the experimenta one for $\gamma = 30$, and $\langle \eta^2 \rangle = 0.81$ (noise of moderate color and large intensity). (b') Comparison between different theoretical predictions for the same parameters as in (b). (c) As for (b) with $\gamma = 6.1$ and $\langle \eta^2 \rangle = 0.04$. (c') As for (b') with the same parameters as (c). (d) As for (b) with $\gamma = 1.54$ and $\langle \eta^2 \rangle = 0.04$ (strongly colored case). (d') As for (b') with the same parameters as (d).

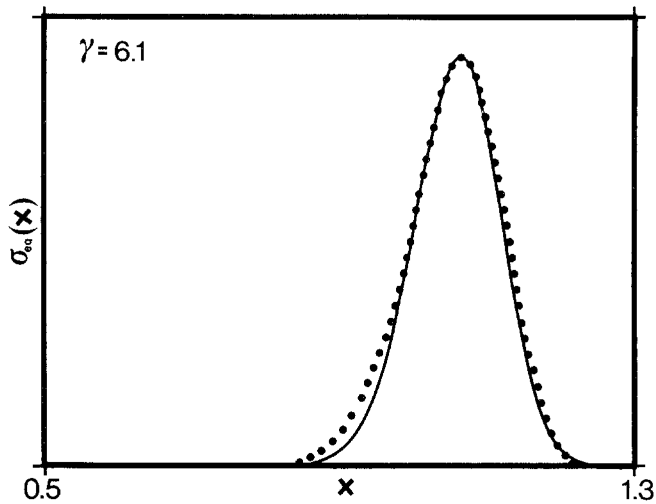


(b)

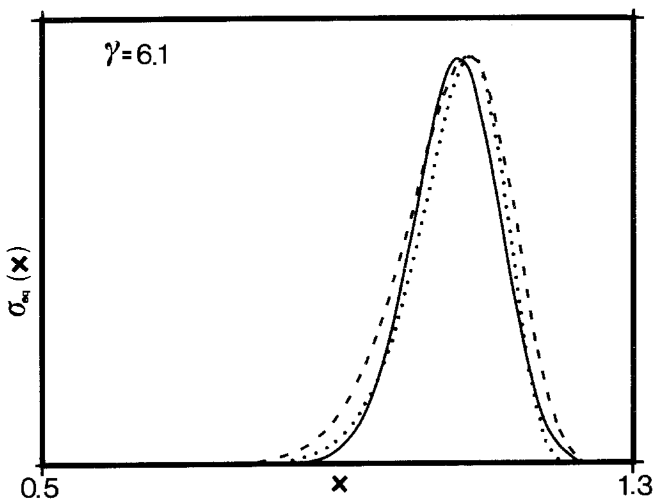


(b′)

Fig. 3 (*continued*)

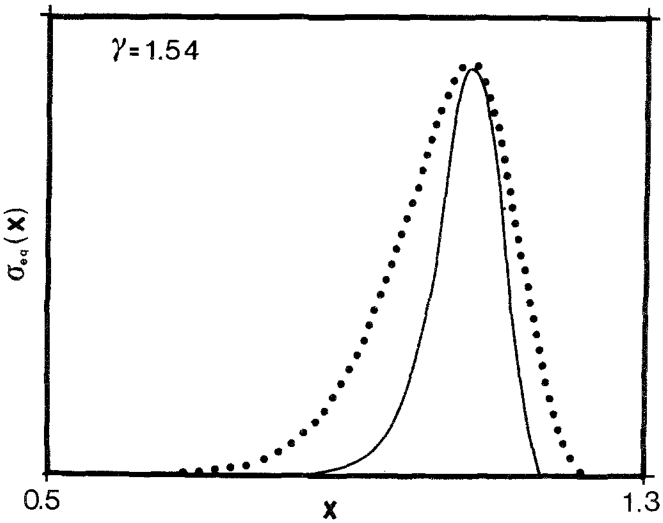


(c)

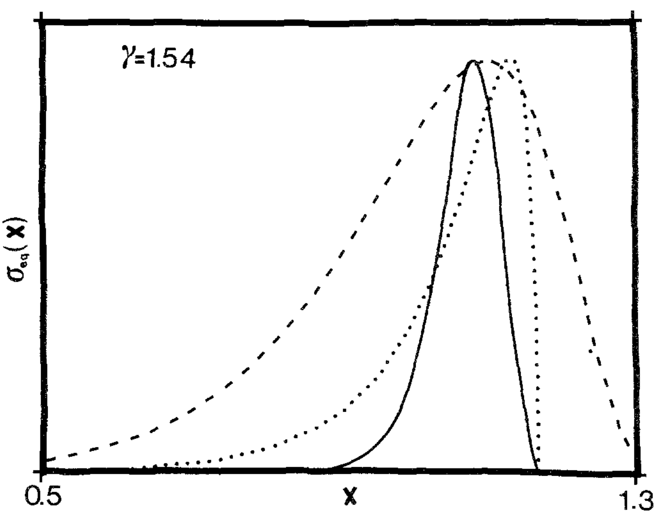


(c′)

Fig. 3 (*continued*)



(d)



(d′)

Fig. 3 (*continued*)

result of our theoretical approach (detailed in Sections 3.2 and 4.3) in the case of the moderately colored noise of Figs. 3b and 3c virtually coincides witht the results of analog simulation. We are thus in a position to show (Figs. 3b′ and 3c′) that the BFPA equation (dashed line) does not provide an agreement with this virtually exact result (solid line) more satisfactory than the white noise approximation (dotted line). Rather, in Fig. 3b′ the white noise approximation seems to be better than the BFPA equation. Figures 3d and 3d′ illustrate the case of strongly colored noise, where all three theoretical predictions (our theoretical approach, the white noise approximation, and the BFPA equation) are largely violated. Further discussion of Figs. 3b–3c′ can be found in the next sections.

3. HIGHER ORDER EFFECTS

To study the corrections of higher order in the interaction between x and ξ, we use the general result of ref. 2,

$$\frac{\partial}{\partial t}\sigma(x;t)=\left\{-\frac{\partial}{\partial x}\varphi(x)+\int_0^t W(s)\exp[-\mathbb{Q}(x)s]\,ds\right\}\sigma(x;t) \tag{3.1}$$

where

$$\exp[-\mathbb{Q}(x)s]\equiv(\exp\mathscr{L})\,a^s\exp[-(\mathscr{L}_a+\mathbb{D})s] \tag{3.2}$$

$\mathbb{D}$ denotes the diffusional operator, which we plan to built up, and

$$\begin{aligned}W(s_0)=&\,\rho_{\mathrm{eq}}^{-1}(\eta)\,P\mathscr{L}_1 e^{(\mathscr{L}_a+\mathscr{L}_b)s_0}(1-P)\,\mathscr{L}_1 P\rho_{\mathrm{eq}}(\eta)\,e^{-\mathscr{L}_a s_0}\\&+\rho_{\mathrm{eq}}^{-1}(\eta)\sum_{n=1}^{\infty}\int_0^{s_0}ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{n-1}}ds_n\\&\times P\mathscr{L}_1 e^{(\mathscr{L}_a+\mathscr{L}_b)s_0}(1-P)\,\mathscr{L}_1(s_1)\cdots\mathscr{L}_1(s_n)\\&\times(1-P)\,\mathscr{L}_1 P\rho_{\mathrm{eq}}(\eta)\,e^{-\mathscr{L}_a s_0}\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mathscr{L}_a&\equiv-\frac{\partial}{\partial x}\varphi(x)\\ \mathscr{L}_b&\equiv\gamma\left(\frac{\partial}{\partial\eta}\eta+\langle\eta^2\rangle\frac{\partial^2}{\partial\eta^2}\right)\\ \mathscr{L}_1&\equiv-\xi\frac{\partial}{\partial x}\psi(x)\end{aligned} \tag{3.4}$$

and the projection operator P is defined by

$$P\rho(x, \eta; t) = \rho_{\text{eq}}(\eta) \int d\eta\, \rho(x, \eta; t) \equiv \rho_{\text{eq}}(\eta)\, \sigma(x; t) \tag{3.5}$$

where $\rho_{\text{eq}}(\eta)$ is the equilibrium distribution of the "bath" defined by

$$\mathscr{L}_b \rho_{\text{eq}}(\eta) = 0 \tag{3.6}$$

3.1. Nonlinear Gaussian Case

In this case we limit ourselves to studying the third order in $\mathscr{L}_1$, which turns out to be the first nonvanishing contribution to the BFPA. Since both the lowerst order contribution to $\mathbb{D}$ and that to $W(s)$ are of second order $\mathscr{L}_1$, $\mathbb{D}$ begins exerting its influence on its own expression [see Eqs. (3.1)–(3.3)] at the fourth order in $\mathscr{L}_1$. Therefore we neglect $\mathbb{D}$ appearing in Eq. (3.2), and rewrite Eq. (3.1) as

$$\frac{\partial}{\partial t} \sigma(x; t) = \left[-\frac{\partial}{\partial x} \varphi(x) + \int_0^t W(s)\, ds \right] \sigma(x; t) \tag{3.7}$$

where

$$W(s) = W_2(s) + W_3(s) \tag{3.8}$$

$W_2(s)$ and $W_3(s)$ are the second- and third-order terms of the expansion of Eq. (3.3).

$W_2(s)$ coincides with the approximation leading to the BFPA equation. Let us focus our attention on $W_3(s)$. This can be written as follows ($s_0 \equiv s$):

$$\begin{aligned} W_3(s_0) = \int_0^{s_0} ds_1\, X(s_0, s_1) \frac{\partial}{\partial x} \psi(x)\, e^{\mathscr{L}_a(s_0 - s_1)} \\ \times \frac{\partial}{\partial x} \psi(x)\, e^{\mathscr{L}_a s_1} \frac{\partial}{\partial x} \psi(x)\, e^{-\mathscr{L}_a s_0} \end{aligned} \tag{3.9}$$

where

$$X(s_0, s_1) = \rho_{\text{eq}}^{-1}(\eta)\, P\xi e^{\mathscr{L}_b(s_0 - s_1)}\, \xi e^{\mathscr{L}_b s_1}\, \xi P \rho_{\text{eq}}(\eta) \tag{3.10}$$

To calculate the three-times correlation function of Eq. (3.10), it is convenient to follow the following calculation rules:

(a) We apply the time evolution operator $\mathscr{L}_b$ on the left. This means that we are dealing with the operator $\mathscr{L}_b^+$ adjoint to the operator $\mathscr{L}_b$

driving the motion of the variable η of Eq. (2.2). In the case under study in this subsection η is a Gaussian variable and the operator $\mathscr{L}_b^+$ reads

$$\mathscr{L}_b^+ = -\gamma\left(\eta\frac{\partial}{\partial\eta} - \langle\eta^2\rangle\frac{\partial^2}{\partial\eta^2}\right) \tag{3.11}$$

(b) We have to express ξ and ξ^2, with ξ being given by (2.3a), in terms of the proper eigenstates of the operator $\mathscr{L}_b^+$. Since ξ and ξ^2 involve η, η^2, η^3 and η, η^2, η^3, η^4, respectively, we must replace η, η^2, η^3, and η^4 with those suitable linear combinations of η, η^2, η^3, and η^4 that turn out to be the eigenstates of $\mathscr{L}_b^+$. Via a straightforward calculation, it is shown that

$$\begin{gathered}
\mathscr{L}_b^+\eta = -\gamma\eta \\
\mathscr{L}_b^+(\eta^2 - \langle\eta^2\rangle) = -2\gamma(\eta^2 - \langle\eta^2\rangle) \\
\mathscr{L}_b^+(\eta^3 - 3\eta\langle\eta^2\rangle) = -3\gamma(\eta^3 - 3\eta\langle\eta^2\rangle) \\
\mathscr{L}_b^+(\eta^4 - 6\eta^2\langle\eta^2\rangle + 3\langle\eta^2\rangle^2) = -4\gamma(\eta^4 - 6\eta^2\langle\eta^2\rangle + 3\langle\eta^2\rangle^2)
\end{gathered} \tag{3.12}$$

(The first, trivial, equation of this set of equations is used also in the calculation of the term W_2). By applying these calculation rules we obtain

$$\begin{aligned}
X(s_0, s_1) = 8\langle\eta^2\rangle^2\,\{h^2[e^{-\gamma s_0} + e^{-\gamma(s_0+s_1)} \\
+ e^{-\gamma(2s_0-s_1)}] + \langle\eta^2\rangle\, e^{-2\gamma s_0}\}
\end{aligned} \tag{3.13}$$

On the other hand, by using the important relation

$$e^{\mathscr{L}_a s}\frac{\partial}{\partial x}\psi = \frac{\partial}{\partial x}\psi e^{\mathscr{L}_a s}e^{\Pi s} \tag{3.14}$$

[see Eq. (3.5) of ref 2] we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x}\psi(x)\, e^{\mathscr{L}_a(s_0-s_1)}\frac{\partial}{\partial x}\psi(x)\, e^{\mathscr{L}_a s_1}\frac{\partial}{\partial x}\psi(x)\, e^{\mathscr{L}_a s_0} \\
&= \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\, e^{\Pi(s_0-s_1)}\frac{\partial}{\partial x}\psi(x)\, e^{\Pi s_0} \\
&= \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\, e^{\Pi(2s_0-s_1)} \\
&\quad - (s_0 - s_1)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\, \Pi^{(1)\prime}(0) \\
&\quad \times e^{\Pi s_0}\, e^{\Pi(1)(s_0-s_1)}
\end{aligned} \tag{3.15}$$

[The last inequality has been obtained by using Eq. (4.13) of ref. 2.] Note that

$$\Pi^{(1)'}(0) = \frac{3kx^2}{\psi(x)} - \frac{kx^3}{\psi^2(x)}\psi'(x) \tag{3.16}$$

When putting Eqs. (3.13) and (3.15) into Eq. (3.9) and integrating $W_3(s_0)$ over s_0 (after making the preliminary integration over s_1), we obtain

$$\begin{aligned}
&\int_0^\infty W_3(s_0)\,ds_0 \\
&\quad = 8\langle\eta^2\rangle^2 \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\left\{h^2\left[\mathbb{A}(\gamma, x)\,\mathbb{B}(\gamma, x)\right.\right. \\
&\qquad \left.\left. + \mathbb{A}(\gamma, x)\,\mathbb{B}(2\gamma, x) + \frac{1}{2}\mathbb{B}^2(\gamma, x\right] + \frac{\langle\eta^2\rangle}{2}\mathbb{B}(\gamma, x)\,\mathbb{B}(2\gamma, x)\right\} \\
&\qquad - 8\langle\eta^2\rangle^2 \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\left(\frac{3kx^2}{\psi(x)} - \frac{kx^3\psi'(x)}{\psi^2(x)}\right)\psi(x) \\
&\qquad \otimes \{h^2[\mathbb{B}^2(\gamma - \Pi(1), x)\,\mathbb{B}(\gamma, x) + \mathbb{B}^2(\gamma - \Pi(1), x)\,\mathbb{B}(2\gamma, x) \\
&\qquad + \mathbb{B}^2(2\gamma - \Pi(1), x)\,\mathbb{B}(\gamma, x)] + \langle\eta^2\rangle\,\mathbb{B}^2(2\gamma - \Pi(1), x)\,\mathbb{B}(2\gamma, x)\}
\end{aligned} \tag{3.17}$$

where

$$\mathbb{A}(\gamma, x) \equiv \frac{1}{\gamma}\left(1 + \frac{2kx^3}{\psi(x)(\gamma + 4A)}\right) \tag{3.18}$$

$$\mathbb{B}(\tilde{\gamma}, x) \equiv \frac{1}{\gamma}\left(1 + \frac{kx^3}{\psi(x)(\tilde{\gamma} + 2A)}\right) \tag{3.19}$$

When $\tilde{\gamma} = \gamma - \Pi(1)$, $\mathbb{B}(\tilde{\gamma}, x)$ must be determined by having recourse to the hierarchy of functions of Eq. (4.15) of ref. 2.

3.2. Dichotomic Case

In the purely dichotomic case (see Section 2.1), $K(s)$ can be replaced by $K_2(s)$ with no approximation at all. The $(1 - P)\,\mathscr{L}_1 P$ means indeed that the "excitation" term applied to the "ground" state $\rho_{eq}(\tilde{\xi})$ produces a transition to the space spanned by the "excited states. When only one "excited" state is available (this is precisely the case of the dichotomic noise), the excitation $\mathscr{L}_1$ cannot do anything but recover the "ground" state, resulting in

$$(1-P)\,\mathscr{L}_1(1-P)\,\mathscr{L}_1 P=0 \tag{3.20}$$

Thus, from Eq, (3.1) we obtain in this case

$$\frac{\partial}{\partial t}\sigma(x;t)=\mathscr{L}_a\sigma(x;t)+\left[\int_0^t W_2(s)\exp(-\mathbb{Q}s)\,ds\right]\sigma(x;t) \tag{3.21}$$

Thus, the only possible correction to the BFPA equation stems from the expansion of the exponential $\exp(-\mathbb{Q}s)$ of Eq. (3.30) conveniently adapted to the case $\psi'(x)\neq 0$, i.e.,

$$\exp(-\mathbb{Q}s)=\exp\left(-\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\frac{\psi(x)s}{(\gamma-3\Pi)}\right) \tag{3.22}$$

By replacing the exponential, expanded into a Taylor power series up to the first order in s, into the second term of the rhs of Eq. (3.21) and making it tend to infinity, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\sigma(x;t)=\mathscr{L}_a\sigma(x;t)&+\langle\xi^2\rangle_{\text{eq}}\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{\gamma-\Pi}\\&-(\langle\xi^2\rangle_{\text{eq}})^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{(\gamma-\Pi)^2}\\&\times\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{\gamma-3\Pi}\end{aligned} \tag{3.23}$$

The fourth-order corrections that are proven to coincide with the exact results illustrated in Figs. 1 were obtained by putting $\Pi=0$ in the third term on the rhs of Eq. (3.23), replacing $(\gamma-\Pi)^{-1}$ appearing in the second term on the rhs of Eq. (3.23) with $\gamma^{-1}(1+\tau\Pi^{(1)})$, and adopting the renormalization technique of refs. 2 and 12. The motivations behind this approximation are the same as those mentioned in Section 3.1. Details on this satisfactory theoretical result are given in Section 4.1.

4. A THEORETICAL ANALYSIS OF THE BREAKDOWN OF THE LINEAR RESPONSE APPROXIMATION

The major of this section is to show with purely analytical arguments that upon increase of the correlation time τ the linear response approximation breaks down. Let us first focus on the case of dichotomic noise.

4.1. Dichotomic Noise

From Horsthemke and Lefever[13] we see that the equilibrium distribution of the variable x can be written as

$$\sigma_{eq}(x) = \frac{N\psi(x)}{4h^2\,\delta^2\psi^2(x) - \varphi^2(x)} \otimes \exp\left[-\frac{\gamma}{2}\int dx' \left(\frac{1}{\varphi(x') + 2h\,\delta\psi(x')} + \frac{1}{\varphi(x') - 2h\,\delta\psi(x')}\right)\right] \tag{4.1}$$

Let us assume that this equilibrium distribution can be regarded as being the solution of the effective Fokker–Planck equation

$$\frac{\partial}{\partial t}\sigma(x;t) = \left[-\frac{\partial}{\partial x}\varphi(x) + D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\Phi_{\text{eff}}(x)\right]\sigma(x;t) \tag{4.2}$$

The equilibrium distribution of this Fokker–Planck equation reads

$$\sigma_{eq}(x) = \frac{N}{\Phi_{\text{eff}}(x)}\exp\int\frac{\varphi(x')\,dx'}{D\psi(x')\,\Phi_{\text{eff}}(x')} \tag{4.3}$$

where $D = 4h^2\,\delta^2/\gamma$. By comparing Eq. (4.3) with Eq. (4.1), we find that these two distributions coincide with each other if

$$\Phi_{\text{eff}}(x) = \psi(x) - \varphi^2(x)\tau/D\psi(x) \tag{4.4}$$

which must be compared with the corresponding result of the best Fokker–Planck approximation [see Eq. (2.8), here rewritten for the reader's convenience]

$$\Phi(x) = \psi(x) + kx^3\tau/(1 + 2A\tau) \tag{4.5}$$

The astonishing result is that if $\tau \neq 0$, upon decrease of D a larger and larger disagreement between $\Phi_{\text{eff}}(x)$ and $\Phi(x)$ is produced, this being precisely the behavior opposite to that predicted by the linear response approximation. *The linear response approximation leads to an exact result in the white noise limit but is completely invalid for colored noise.* This explains why, as illustrated by Fig. 2, the BFPA turns out to be less satisfactory than the white noise approximation.

We are now indeed in a position to show that the breakdown of the linear response approximation occurs in this case at the first order in the noise correlation time τ. This means that the nonstandard diffusion operator of Eq. (3.23) produces a relevant correction to the BFPA equation at the order $D\tau$. To show that, we have recourse to Eq. (3.23). Neglecting the terms of order higher than τ, we can rewrite Eq. (3.23) as

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[-\frac{\partial}{\partial x}\varphi(x)+D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\tilde{\Phi}(x)\right.$$
$$\left.-D^2\tau\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\right]\sigma(x;t) \qquad (4.6)$$

where $\tilde{\Phi}(x)$ is the approximation at first order in τ to Eq. (4.5). Note that the equilibrium distribution of x in the white noise limit can be rewritten as

$$\sigma_{\text{eq}}^{(w)}(x)=\frac{N}{\psi(x)}\exp\int\frac{\varphi(x')\,dx'}{D\psi^2(x')} \qquad (4.7)$$

from which we obtain

$$\frac{\partial}{\partial x}\sigma_{\text{eq}}^{(w)}(x)=A(x)\,\sigma_{\text{eq}}^{(w)}(x) \qquad (4.8)$$

where

$$A(x)\equiv\frac{\varphi(x)-D\psi(x)\,\psi'(x)}{D\psi^2} \qquad (4.8')$$

Differentiating again both terms of Eq. (4.8), we obtain

$$\frac{\partial^2}{\partial x^2}\sigma_{\text{eq}}^{(w)}(x)=B(x)\,\sigma_{\text{eq}}^{(w)}(x) \qquad (4.9)$$

where

$$B(x)\equiv\frac{\varphi^2(x)-D[4\psi(x)\,\psi'(x)\,\varphi(x)+\psi^2(x)\,\varphi'(x)]+2D^2\psi^2(x)\,\psi'^2(x)}{D\psi^4(x)} \qquad (4.9')$$

The result expressed via the set of equations from (4.8) to (4.9′) allows us to replace the last term on the rhs of Eq. (4.6) with an effective diffusion term of standard type. This can be done by writing

$$D^2\tau^2 \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x)\, \sigma(x; t)$$

$$= D^2\tau^2 \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x)$$

$$\times \left[\psi'(x)\, \sigma(x; t) + \psi(x) \frac{\partial}{\partial x} \sigma(x; t) \right]$$

$$= D^2\tau^2 \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x) \Big\{ [(\psi'(x))^2 + \psi(x)\, \psi''(x)]\, \sigma(x; t)$$

$$+ 3\psi(x)\, \psi'(x) \frac{\partial}{\partial x} \sigma(x; t) + \psi^2(x) \frac{\partial^2}{\partial x^2} \sigma(x; t) \Big\} \tag{4.10}$$

and assuming that

$$\frac{\partial}{\partial x} \sigma(x; t) = A(x)\, \sigma(x; t) \tag{4.11}$$

$$\frac{\partial^2}{\partial x^2} (\sigma(x; t) = B(x)\, \sigma(x; t) \tag{4.12}$$

This assumption implies that the system is not very far from the equilibrium distribution of the white noise limit, where Eqs. (4.11) and (4.12) would coincide with the exact relations (4.8) and (4.9). Then we replace Eqs. (4.11) and (4.12) [with $A(x)$ and $B(x)$ defined by Eqs. (4.8′) and (4.9′)] into Eq. (4.10). The resulting expression, in turn, is replaced into the last term on the rhs of Eq. (4.6). Note that this procedure allows us to replace the BFPA diffusion function $\tilde{\Phi}(x)$ by the renormalized one:

$$\Phi_{\text{renorm}}(x) = [kx^3 + \psi'(x)\, \varphi(x) - \varphi'(x)\, \psi(x)]\,\tau$$

$$+ \psi(x) - \frac{\varphi^2(x)\,\tau}{D\psi(x)} \tag{4.13}$$

Note that the first term on the rhs of Eq. (4.13) exactly vanishes [as can be easly seen by making explicit the term $\psi'(x)\, \varphi(x) - \varphi'(x)\, \psi(x)$ via Eqs. (2.3b) and (2.3e) and using the definition of Eq. (2.8′)], thereby producing

$$\Phi_{\text{renorm}}(x) = \psi(x) - \frac{\varphi^2(x)\,\tau}{D\psi(x)} \tag{4.14}$$

We thus see that $\Phi_{\text{renorm}}(x)$ coincides with $\Phi_{\text{eff}}(x)$ [see Eq. (4.4)].

It must be stressed that in this special case, by expanding $\Phi(x)$ of the BFPA equation up to the first order in τ and having recourse to the perturbation contribution of order $\mathscr{L}_1^4$ [with the respective function of type $\Phi(x)$ truncated at the order τ^0] we recover the exact result of Eq. (4.4). Note that this is a lucky consequence of the fact that the exact solution [Eq. (4.4)] is a linear function of τ. The terms of order $\mathscr{L}_1^6$ are expected to cancel with the terms of order $\mathscr{L}_1^4$ and $\mathscr{L}_1^2$ with the respective functions of type $\Phi(x)$ expanded up to orders τ and τ^2, respectively. The result of Eq. (4.14) strongly supports the validity of the assumptions of Eqs. (4.11) and (4.12). Figure 2 illustrates the discrepancy between the results of the BFPA equation and the result of the above renormalization procedure, which leads to an equilibrium distribution coincident with the exact one.

4.2. Nonlinear Gaussian Noise

The analysis of this case is similar to that of the case of the dichotomic noise. Even in this case the first nonvanishing diffusion term of nonstandard type is characterized by a third-order derivative [see Eq. (3.17)]. The functions $\mathbb{A}(\tilde{\gamma}, x)$ and $\mathbb{B}(\tilde{\gamma}, x)$ of Eqs. (3.18) and (3.19) are replaced by $1/\gamma$, and the second term on the rhs of Eq. (3.17), being of higher order in τ, is disregarded. This approximation is believed to produce no significant effect (for the negligible effect of color at the perturbation order $\mathscr{L}_1^2$ see Fig. 3b′). The remaining third-order derivative is then replaced by the equivalent second-order one by using the procedure detailed in Section 4.1. The resulting diffusion operator is then comparable with the perturbation term at the second order in $\mathscr{L}_1$, with $\Phi(x)$ replaced with its white noise approximation. This shows that the perturbation term at the third order in $\mathscr{L}_1$ provokes a breakdown of the linear response approximation affecting also the white noise approximation (the most severe breakdown of the linear response possible!). The reason the white noise approximation of ref. 10 does not result in a large disagreement with the actual equilibrium distribution at moderate values of the parameter $\langle \eta^2 \rangle/\gamma$ (see Figs. 3b′ and 3c′) is that the correction stemming from $\mathscr{L}_1^3$ mainly concerns the tails of the equilibrium x distribution, resulting in large effects only in the x region characterized by a virtually vanishing population. This is clearly illustrated via Fig. 4. If the noise intensity become so large as to populate at a significant extent the large-x regions, the white noise approximation of Sagues and San Miguel[10] would break down also in the limiting case $\gamma \to \infty$ (see Fig. 4).

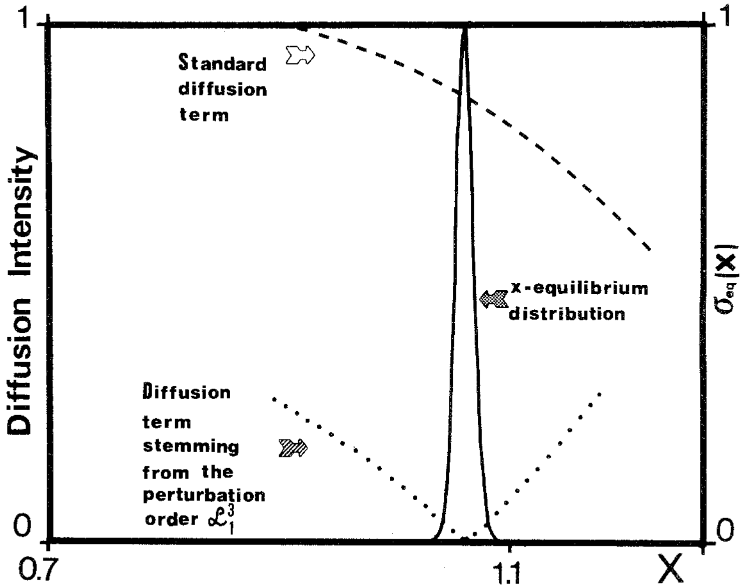


Fig. 4. Comparison between (- -) the absolute value of the diffusion function $D_1\Phi(x)+D_2\Phi_2(x)$ of the BFPA equation for the nonlinear Gaussian noise [see Eqs. (2.16) and (2.16′)] and ($\cdots$) the absolute value of the diffusion term obtained by replacing the third-order derivative of Eq. (3.17) with an equivalent one of second order. Both diffusion coefficients are plotted as functions of x. In order to make it clear that the correction diffusion term virtually vanishes in the most populated region, the theoretical equilibrium distribution of the variable x is also plotted as a function of x (solid line). It is thus shown that the correction term plays a negligible role in the case of noise of weak intensity (distributions as in Fig. 3a), while becoming important in the case of noise of large intensity (distributions as in Fig. 3b′). The corresponding ordinate scale (in arbitrary units) is on the right. The parameters of Eq. (2.1) are $\Gamma=1$, $\gamma=500$, $h=1.5$, $\langle\eta^2\rangle=0.04$, $\alpha=1$, $\beta=1/2$. Note that the ratio of the value of the former to the latter diffusion coefficients is virtually independent of γ as $\gamma>200$. Thus, the third-order ($\mathscr{L}_1^3$) diffusion term is correct also in the white-noise limit.

5. CONCLUDING REMARKS

This paper shows that the linear response approximation leading to the BFPA, i.e., Eq. (3.8) of ref. 2, can be applied also to non-Gaussian statistics. In this case, however, the range of validity of this approximation is still more limited than in the Gaussian case of ref. 2. In the case of dichotomic noise, the BFPA breaks down at the first order in the correlation time τ of the noise ξ. In the case of the quadratic Gaussian noise the breakdown takes place at the zeroth order in τ. This is a dramatic condition where the Fokker-Planck approximation is completely invalid.

Figure 3a illustrates the case where, due to a suitable choice of the parameters, this catastrophic condition is not evident. However, the corresponding theoretical analysis of Section 4.2 shows that systems of the type of Eq. (2.1′) can be found where this breakdown at $\gamma = \infty$ might produce impressive effects. In the case illustrated in Fig. 4 this breakdown would be rendered evident by increasing the population of the large-x regions (thereby increasing the noise intensity). With different forms of $\psi(x)$, noises of large intensity might not be necessary.

It must be pointed out that the theoretical results of the present paper have been made possible by using the projection method, which, not being restricted to the Gaussian case, allowed us to study the key case of the dichotomic noise.

A further remarkable result of this paper is the excellent agreement between the theory and analog simulation (Figs. 1 and 2). This excellent agreement implies both the validity of the procedure adopted to replace third- and fourth-order x derivatives with equivalent standard diffusion terms and the very good accuracy of our analog simulation apparatus.

APPENDIX

This Appendix is devoted to illustrating the apparatus behind the results of analog simulation used in this paper to support our theoretical arguments.

We simulated Eq. (2.1) with η either Gaussian or dichotomic by using the electric device illustrated in Fig. 5.

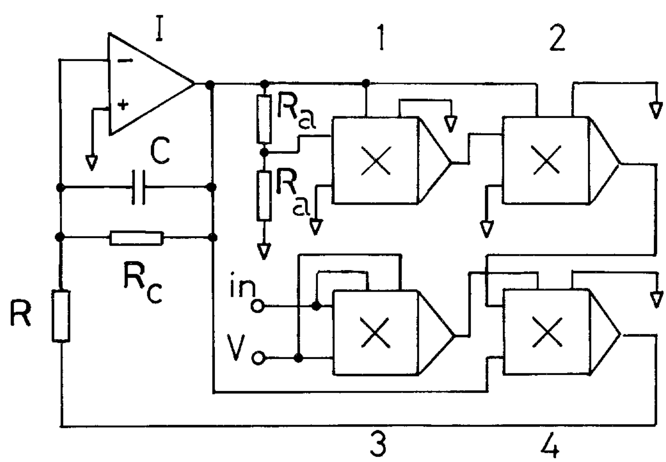


Fig. 5. Scheme of the analog circuit used to simuate Eq. (2.1).

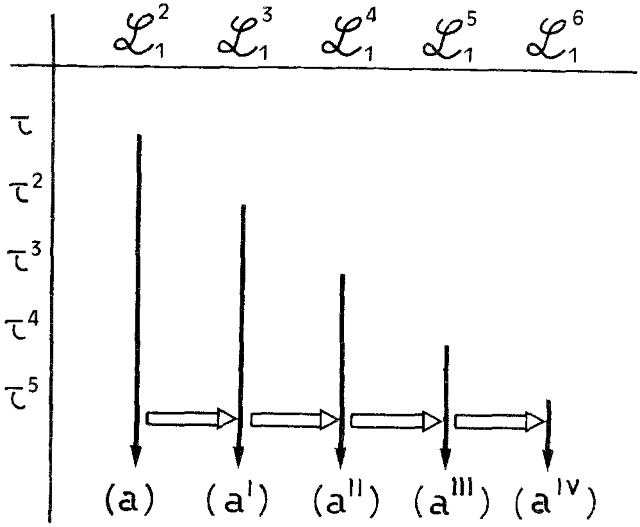


Figure 6

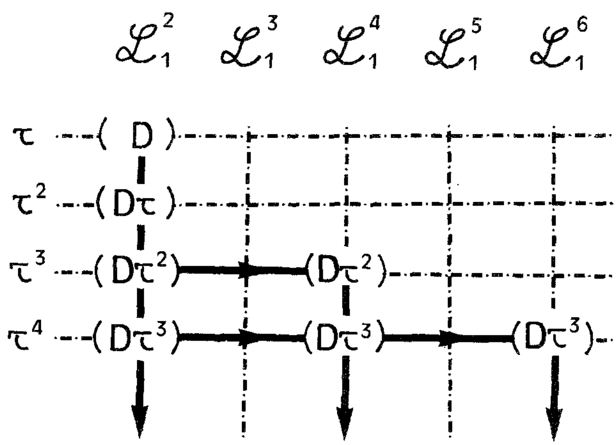


Figure 7

The integration operation is simulated by means of a Miller integrator (I) and the nonlinear terms are simulated via operational multiplier devices (AD 354).

We get the nonlinear noise sending a colored noise to one of the inputs of the multiplier 3.

In the case of Gaussian noise color changes were obtained by changing the characteristic time of a low-pass filter put between the output of the noise generator and the input of our device. A constant voltage is added to the noise at the input of the multiplier 3 to reproduce the deterministic term of Eq. (2.1).

Note that it is convenient to balance the output of the Miller integrator to get a more accurate symmetry of the deterministic system. To test this symmetry, we measure the values of the two stable equilibrium points in the stationary condition and in the absence of external noise. The output voltage is sent to a computer to have a statistical distribution.

It has to be stressed that, although crossing the zero value is rigorously forbidden by the theory, in the analog experiment this may take place as a consequence of the additive internal noise. Luckily this does not influence the results, because this only means that both symmetric solutions are exhibited by our analog system. Moreover, this affords another way to test the symmetry of the system. The accuracy of the analog simulation apparatus is illustrated by comparing the exact solution of the dichotomic case with analog one (see Fig. 1).

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